THE HAUSDORFF DIMENSION OF SELF–AFFINE SIERPIŃSKI SPONGES

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ABSTRACT

We compute the Hausdorff dimension of limit sets generated by 3-dimensional self-affine mappings with diagonal matrices of the form

$$A_{ijk} = \begin{pmatrix} a_{ijk} & 0 & 0 \\ 0 & b_{ij} & 0 \\ 0 & 0 & c_i \end{pmatrix},$$

where $0 < a_{ijk} \leq b_{ij} \leq c_i < 1$, and a Markov partition as in Figure 1. By doing so we show that the variational principle for the dimension holds for this class.

Keywords: Hausdorff dimension; variational principle

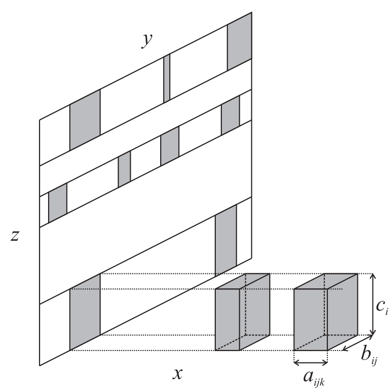

FIGURE 1.

1. INTRODUCTION

A main difficulty in calculating Hausdorff dimension is the phenomenon of *non-conformality* which arises when we have several rates of expansion. In the 1-dimensional (conformal) setting the computation of Hausdorff dimension is possible, at least in the *uniformly expanding* context, due to the thermodynamic formalism introduced by Sinai-Ruelle-Bowen (see [3] and [12]). The problem of calculating Hausdorff dimension in the non-conformal setting was first considered by Bedford [2] and McMullen [10]. They showed independently that for the class of transformations called *general Sierpiński carpets*, there exists an ergodic measure of full Hausdorff dimension. Following these works several extensions have been made, e.g.

in [5], [6], [8], [9] and [1]. The transformations considered in these works are in the plane and the methods used therein present difficulties when extending to higher dimensions. So far the computation of Hausdorff dimension for transformations in higher dimensions was only done in [7] for *d-dimensional general Sierpiński carpets* ($d \geq 2$). In this work we compute the Hausdorff dimension of some self-affine generalizations of 3-dimensional general Sierpiński carpets, which are 3-dimensional extensions of the planar constructions considered in [5].

We try to follow the methods in [5] and [8]. The main difficulty is proving the upper estimate for the Hausdorff dimension of the invariant set, since for each point in the set we must choose carefully an appropriate invariant probability measure. This is done by constructing a two-parameter family of Bernoulli measures, one parameter stands for the Hausdorff dimension of *horizontal fibres*, and the other is a *non-conformality* parameter (see Lemmas 4 and 6). We note that in [5] the authors also use a one-parameter family of Bernoulli measures, with a *non-conformality* parameter. The proof of this upper estimate also depends heavily on a calculus lemma which is proved in Section 4.

We begin by describing what we mean by a *d-general Sierpiński carpet* ($d \geq 2$). Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus and $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be given by

$$f(x_1, x_2, ..., x_d) = (l_1x_1, l_2x_2, ..., l_dx_d)$$

where $l_1 \geq l_2 \geq ... \geq l_d > 1$ are integers. The grids of hyperplanes

$$\begin{aligned} \{i/l_1\} \times [0,1]^{d-1}, i = 0, ..., l_1 - 1 \\ [0,1] \times \{i/l_2\} \times [0,1]^{d-2}, i = 0, ..., l_2 - 1 \\ \vdots \\ [0,1]^{d-1} \times \{i/l_d\}, i = 0, ..., l_d - 1 \end{aligned}$$

form a set of boxes each of which is mapped by f onto the entire torus (these boxes are the domains of invertibility of f). Now choose some of these boxes and consider the fractal set Λ consisting of those points that always remain in these chosen boxes when iterating f. Geometrically, Λ is the limit (in the Hausdorff metric), or the intersection, of *n-approximations*: the 1-approximation consists of the chosen boxes, the 2-approximation consists in replacing each box of the 1-aproximation by a rescaled copy of the 1-approximation performed by an affine map of diagonal matrix, and so on. We say that Λ is a *d-general Sierpiński carpet.* The Hausdorff dimension of d-general Sierpiński carpets was computed in [7], where the authors prove the existence of an invariant Bernoulli measure of full dimension.

Now we introduce a class of sets in $\mathbb{R}^d$ that are self-affine generalizations of d-general Sierpiński carpets. Let $S_1, S_2, ..., S_r$ be contractions of $\mathbb{R}^d$. Then there is a unique nonempty compact set Λ of $\mathbb{R}^d$ such that

$$\Lambda = \bigcup_{i=1}^{r} S_i(\Lambda).$$

We will refer to Λ as the limit set of the semigroup generated by $S_1, S_2, ..., S_r$. We are going to consider sets Λ which are limit sets of the semigroup generated by the d-dimensional mappings $A_{i^1 i^2 \ldots i^d}$ given by

$$A_{i^1 i^2 \ldots i^d} = \begin{pmatrix} a_{i^1 i^2 \ldots i^d} & 0 & \cdots & 0 & \\ 0 & a_{i^1 i^2 \ldots i^{d-1}} & 0 & \cdots & 0 \\ & \cdots & & & \\ 0 & \cdots & 0 & a_{i^1} & \end{pmatrix} x + \begin{pmatrix} u_{i^1 i^2 \ldots i^d} \\ u_{i^1 i^2 \ldots i^{d-1}} \\ \vdots \\ u_{i^1} \end{pmatrix}$$

for $(i^1, i^2, ..., i^d) \in \mathcal{I}$. Here

$$\mathcal{I} = \{(i^1, i^2, ..., i^d) : 1 \le i^1 \le m\,,\ 1 \le i^2 \le m_{i^1},$$
$$1 \le i^3 \le m_{i^1 i^2}\,, ..., 1 \le i^d \le m_{i^1 i^2 \ldots i^{d-1}}\}$$

is a finite index set, and $0 < a_{i^1 \ldots i^k} < 1$, $k = 1, ..., d$ satisfy

$$a_{i^1 \ldots i^k i^{k+1}} \le a_{i^1 \ldots i^k}\,.$$

Also, for each $(i^1, ..., i^d) \in \mathcal{I}$ and $k \in \{1, ..., d\}$,

$$\sum_{i^k=1}^{m_{i^1 \ldots i^{k-1}}} a_{i^1 \ldots i^k} \le 1$$

(by convention, when $k = 1$ the end of the sum is m) and

$$0 \le u_{i^1 \ldots i^k} < u_{i^1 \ldots i^k+1} < 1\,, \quad u_{i^1 \ldots i^k+1} - u_{i^1 \ldots i^k} \ge a_{i^1 \ldots i^k}\,,$$

when $k > 1$, $i^k = m_{i^1 \ldots i^{k-1}}$ we substitute $u_{i^1 \ldots i^k+1}$ by 1. These hypotheses guarantee that the boxes

$$R_{i^1 \ldots i^d} = A_{i^4 \ldots i^d}([0,1]^d)$$

have interiors that are pairwise disjoint, with edges parallel to the coordinate axes, the box $R_{i^1 \ldots i^d}$ having $k^{th}-$edge with length $a_{i^1 \ldots i^k}$. Geometrically, Λ is constructed like the d-general Sierpiński carpets, with the 1-approximation consisting of the boxes $R_{i^1 \ldots i^d}$, the 2-approximation consisting in replacing each box of the 1-aproximation by an affine copy of the 2-approximation, and so on. See Figure 1 for an illustration of the case $d = 3$. The case $d = 2$ considered in [5] and [8] corresponds to the projection onto the yz-plane of Figure 1.

Definition 1. When $d = 3$ we say that Λ is a *self-affine Sierpiński sponge.*

In this case we also use the notation $i = i^1, j = i^2, k = i^3$, $c_i = a_i$ and $b_{ij} = a_{ij}$. We will need the following *generic* hypothesis on the numbers a_{ijk}. For each $t \in [0,1]$, there exist $1 \le i \le m$ and $1 \le j < j' \le m_i$ such that

$$\sum_{k=1}^{m_{ij}} a_{ijk}^t \ne \sum_{k=1}^{m_{ij'}} a_{ij'k}^t. \tag{1}$$

Notation: $\dim_{\mathrm{H}} \Lambda$ stands for the Hausdorff dimension of a set Λ.

Theorem A. *Let $d = 3$ and suppose (1) is satisfied. Then*

$$\dim_{\mathrm{H}} \Lambda = \sup_{\mathbf{p}} \{\lambda(\mathbf{p}) + t(\mathbf{p})\} \tag{2}$$

where $\mathbf{p} = (p_{i^1 \ldots i^{d-1}})$ is a collection of non-negative numbers satisfying

$$p_{i^1 \ldots i^k} = \sum_{i^{k+1}} p_{i^1 \ldots i^k i^{k+1}}\,, \quad k = d-2, d-3, ..., 1$$

$$\sum_{i^1} p_{i^1} = 1,$$

the number $\lambda(\mathbf{p})$ is given by

$$\lambda(\mathbf{p}) = \sum_{k=1}^{d-1} \frac{\sum_{i^1 \ldots i^k} p_{i^1 \ldots i^k} \log p_{i^1 \ldots i^k} - \sum_{i^1 \ldots i^{k-1}} p_{i^1 \ldots i^{k-1}} \log p_{i^1 \ldots i^{k-1}}}{\sum_{i^1 \ldots i^k} p_{i^1 \ldots i^k} \log a_{i^1 \ldots i^k}},$$

(by convention: $0 \log 0 = 0$; for $k = 1$ the second sum in the numerator is 0) and $t(\mathbf{p})$ is the unique real in $[0,1]$ satisfying

$$\sum_{i^1 \ldots i^{d-1}} p_{i^1 \ldots i^{d-1}} \log \left(\sum_{i^d} a_{i^1 \ldots i^{d-1} i^d}^{t(\mathbf{p})} \right) = 0.$$

Remark 1.

(1) We have mantainned the multidimensional notation because some of the results in the proof of Theorem A hold for any $d \geq 2$ (and without hypothesis (1)), namely the inequality $\geq$ in (2).
(2) The number $\lambda(\mathbf{p})$ is the Hausdorff dimension in the hyperplane $x_2 \ldots x_d$ of the set of generic points for the distribution $\mathbf{p}$; the number $t(\mathbf{p})$ is the Hausdorff dimension of a typical 1-dimensional fibre in the x_1-direction relative to the distribution $\mathbf{p}$, and is given by a random Moran formula.

It follows from the proof of Theorem A (see Lemma 3) that the expression between brackets in (2) is the Hausdorff dimension of a Bernoulli measure $\mu_{\mathbf{p}}$. Since the functions $\mathbf{p} \mapsto \lambda(\mathbf{p})$ and $\mathbf{p} \mapsto t(\mathbf{p})$ are continuous, we obtain the following.

Corollary A. *With the same hypotheses of Theorem A, there exists* $\mathbf{p}^*$ *such that*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{H}} \mu_{\mathbf{p}^*}.$$

2. Basic results

Here we mention some basic results about fractal geometry and pointwise dimension. For proofs we refer the reader to the books [4] and [11].

We are going to define the Hausdorff dimension of a set $F \subset \mathbb{R}^n$. The diameter of a set $U \subset \mathbb{R}^n$ is denoted by $|U|$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $|U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. Given $t \geq 0$, we define the *t-dimensional Hausdorff measure of* F as

$$\mathcal{H}^t(F) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^t : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

It is not difficult to see that there is a critical value t_0 such that

$$\mathcal{H}^t(F) = \begin{cases} \infty & \text{if } t < t_0 \\ 0 & \text{if } t > t_0. \end{cases}$$

We define the *Hausdorff dimension* of F, written $\dim_{\mathrm{H}} F$, as being this critical value t_0.

Let μ be a Borel probability measure on $\mathbb{R}^n$. The Hausdorff dimension of the measure μ was defined by L.-S. Young as

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} F : \mu(F) = 1\}.$$

So, by definition, one has

$$\dim_{\mathrm{H}} F \geq \sup\{\dim_{\mathrm{H}} \mu : \mu(F) = 1\}.$$

In this paper we are interested in the validity of the opposite inequality in a dynamical context. In practice, to calculate the Hausdorff dimension of a measure, it is useful to compute its *lower pointwise dimension*:

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ stands for the open ball of radius r centered at the point x. The relations between these dimensions are given by the following propositions.

Proposition 1.

(1) *If* $\underline{d}_\mu(x) \geq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \geq d$.
(2) *If* $\underline{d}_\mu(x) \leq d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu \leq d$.
(3) *If* $\underline{d}_\mu(x) = d$ *for* μ*-a.e.* x *then* $\dim_{\mathrm{H}} \mu = d$.

Proposition 2. *If* $\underline{d}_\mu(x) \leq d$ *for every* $x \in F$ *then* $\dim_{\mathrm{H}} F \leq d$.

3. Proof of Theorem A

Part 1: $\dim_{\mathrm{H}} \Lambda \geq \sup_{\mathbf{p}}\{\lambda(\mathbf{p}) + t(\mathbf{p})\}$

In this part, d is any integer greater than or equal to 2.

There is a natural symbolic representation associated with our system that we shall describe now. Consider the sequence space $\Omega = \mathcal{I}^{\mathbb{N}}$. Elements of Ω will be represented by $\omega = (\omega_1, \omega_2, ...)$ where $\omega_n = (i_n^1, ..., i_n^d) \in \mathcal{I}$. Given $\omega \in \Omega$ and $n \in \mathbb{N}$, let $\omega(n) = (\omega_1, \omega_2, ..., \omega_n)$ and define the *cylinder of order* n,

$$C_{\omega(n)} = \{\omega' \in \Omega : \omega'_l = \omega_l,\, l = 1, ..., n\},$$

and the *basic box of order* n,

$$R_{\omega(n)} = A_{\omega_1} \circ A_{\omega_2} \circ \cdots \circ A_{\omega_n}([0,1]^d).$$

We have that $(R_{\omega(n)})_n$ is a decreasing sequence of closed boxes having $k^{th}-$edge with length $\prod_{l=1}^n a_{i_l^1 \ldots i_l^k}$. Thus $\bigcap_{n=1}^\infty R_{\omega(n)}$ consists of a single point which belongs to Λ that we denote by $\chi(\omega)$. This defines a continuous and surjective map $\chi\colon \Omega \to \Lambda$ which is at most 2^d to 1, and only fails to be a homeomorphism when some of the boxes $R_{i^1 \ldots i^d}$ have nonempty intersection.

We shall construct probability measures $\mu_{\mathbf{p}}$ supported on Λ with

$$\dim_{\mathrm{H}} \mu_{\mathbf{p}} = \lambda(\mathbf{p}) + t(\mathbf{p}).$$

This gives what we want because $\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \mu_{\mathbf{p}}$.

Let $\tilde{\mu}_{\mathbf{p}}$ be the Bernoulli measure on Ω that assigns to each symbol $(i^1, ..., i^d) \in \mathcal{I}$ the probability

$$p_{i^1 \ldots i^{d-1}} \frac{a_{i^1 \ldots i^d}^{t(\mathbf{p})}}{\sum_{j^d} a_{i^1 \ldots i^{d-1} j^d}^{t(\mathbf{p})}}.$$

In other words, we have

$$\tilde{\mu}_{\mathbf{p}}(C_{\omega(n)}) = \prod_{l=1}^n p_{i_l^1 \ldots i_l^{d-1}} \frac{a_{i_l^1 \ldots i_l^d}^{t(\mathbf{p})}}{\sum_{j^d} a_{i_l^1 \ldots i_l^{d-1} j^d}^{t(\mathbf{p})}}.$$

Let $\mu_{\mathbf{p}}$ be the probability measure on Λ which is the pushforward of $\tilde{\mu}_{\mathbf{p}}$ by χ, i.e. $\mu_{\mathbf{p}} = \tilde{\mu}_{\mathbf{p}} \circ \chi^{-1}$.

For calculating the Hausdorff dimension of $\mu_{\mathbf{p}}$, we shall consider some special sets called *approximate cubes*. Given $\omega \in \Omega$ and $n \in \mathbb{N}$ such that $n \geq (\log \min a_{i^1 \ldots i^d})/(\log \max a_{i^1})$, define $L_n^0(\omega) = n$,

$$\begin{aligned} L_n^1(\omega) &= \max\left\{k \geq 1 : \prod_{l=1}^n a_{i_l^1} \leq \prod_{l=1}^k a_{i_l^1 i_l^2}\right\} \\ &\vdots \\ L_n^{d-1}(\omega) &= \max\left\{k \geq 1 : \prod_{l=1}^n a_{i_l^1} \leq \prod_{l=1}^k a_{i_l^1 \ldots i_l^d}\right\} \end{aligned} \tag{3}$$

and the *approximate cube*

$$B_n(\omega) = \left\{ \overline{\omega} \in \Omega : \begin{array}{c} \overline{i_l^1} = i_l^1,\, l = 1, ..., n \\ \overline{i_l^2} = i_l^2,\, l = 1, ..., L_n^1(\omega) \\ \vdots \\ \overline{i_l^d} = i_l^d,\, l = 1, ..., L_n^{d-1}(\omega) \end{array} \right\}.$$

We have that each approximate cube $B_n(\omega)$ is a finite union of cylinder sets, and that approximate cubes are *nested*, i.e., given two, say $B_n(\omega)$ and $B_{n'}(\omega')$, either $B_n(\omega) \cap B_{n'}(\omega') = \emptyset$ or $B_n(\omega) \subset B_{n'}(\omega')$ or $B_{n'}(\omega') \subset B_n(\omega)$. Moreover, $\chi(B_n(\omega)) = \tilde{B}_n(\omega) \cap \Lambda$ where $\tilde{B}_n(\omega)$ is a closed box in $\mathbb{R}^d$ with edges parallel to the coordinate axes, the $k^{th}-$edge with length $\prod_{l=1}^{L_n^{k-1}(\omega)} a_{i_l^1 \dots i_l^k}$. By (3),

$$1 \leq \frac{\prod_{l=1}^{L_n^{k-1}(\omega)} a_{i_l^1 \dots i_l^k}}{\prod_{l=1}^{n} a_{i_l^1}} \leq \max a_{i^1 \dots i^d}^{-1}, \tag{4}$$

for $k = 1, ..., d$, hence the term "approximate cube". It follows from (4) that

$$\frac{\sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1 \dots i_l^k}}{\sum_{l=1}^{n} \log a_{i_l^1}} = 1 + \frac{1}{n} \frac{\sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1 \dots i_l^k} - \sum_{l=1}^{n} \log a_{i_l^1}}{\frac{1}{n}\sum_{l=1}^{n} \log a_{i_l^1}} \to 1. \tag{5}$$

Also observe that $L_n^{k+1}(\omega) \leq L_n^k(\omega)$ and $L_n^k(\omega) \to \infty$ as $n \to \infty$.

First we calculate the dimension of the "vertical" part. Let

$$\mathcal{J} = \{(i^1, ..., i^{d-1}) : (i^1, ..., i^{d-1}, i^d) \in \mathcal{I} \text{ for some } i^d\}$$

and $\Gamma = \mathcal{J}^{\mathbb{N}}$. Consider the natural projections $\tilde{\pi} \colon \Omega \to \Gamma$ and $\pi \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ given by $\pi(x_1, ..., x_d) = (x_2, ..., x_d)$. We consider the measures

$$\tilde{\nu}_{\mathbf{p}} = \tilde{\mu}_{\mathbf{p}} \circ \tilde{\pi}^{-1} \quad \text{and} \quad \nu_{\mathbf{p}} = \mu_{\mathbf{p}} \circ \pi^{-1}.$$

Lemma 1. *If $d > 2$ then for every $k \in \{1, ..., d-2\}$,*

$$\frac{L_n^k(\omega)}{L_n^{k-1}(\omega)} \to \frac{\sum_{i^1 \dots i^k} p_{i^1 \dots i^k} \log a_{i^1 \dots i^k}}{\sum_{i^1 \dots i^{k+1}} p_{i^1 \dots i^{k+1}} \log a_{i^1 \dots i^{k+1}}} \text{ for } \tilde{\nu}_{\mathbf{p}}\text{-a.e. } \omega.$$

Proof. It follows from (5) that

$$\frac{\sum_{l=1}^{L_n^k(\omega)} \log a_{i_l^1 \dots i_l^{k+1}}}{\sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1 \dots i_l^k}} \to 1. \tag{6}$$

By Kolmogorov's Strong Law of Large Numbers (KSLLN),

$$\frac{1}{L_n^{k-1}(\omega)} \sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1 \dots i_l^k} \to \sum_{i^1 \dots i^k} p_{i^1 \dots i^k} \log a_{i^1 \dots i^k} \text{ for } \tilde{\nu}_{\mathbf{p}}\text{-a.e. } \omega \tag{7}$$

and (redundantly)

$$\frac{1}{L_n^k(\omega)} \sum_{l=4}^{L_n^k(\omega)} \log a_{i_l^1 \dots i_l^{k+1}} \to \sum_{i^1 \dots i^{k+1}} p_{i^1 \dots i^{k+1}} \log a_{i^1 \dots i^{k+1}} \text{ for } \tilde{\nu}_{\mathbf{p}}\text{-a.e. } \omega. \tag{8}$$

The result follows by (6), (7) and (8). $\square$

The next lemma is a multidimensional version of [5, Proposition 3.3].

Lemma 2. $\dim_{\mathrm{H}} \nu_{\mathbf{p}} = \lambda(\mathbf{p})$.

Proof. To calculate the Hausdorff dimension of $\nu_{\mathbf{p}}$ we are going calculate its pointwise dimension and use Proposition 1. Remember that $\chi(B_n(\omega)) = \tilde{B}_n(\omega) \cap \Lambda$ where, by (4), $\tilde{B}_n(\omega)$ is "approximately" a ball in $\mathbb{R}^d$ with radius $\prod_{l=1}^{n} a_{i_l^1}$, and that

$$\nu_{\mathbf{p}}(\pi \tilde{B}_n(\omega)) = \tilde{\nu}_{\mathbf{p}}(\tilde{\pi} B_n(\omega)).$$

Also, χ is at most 2^d to 1. Taking this into account, by Proposition 1 together with [11, Theorem 15.3], one is left to prove that

$$\lim_{n\to\infty} \frac{\log \tilde\nu_{\mathbf{p}}(\tilde\pi B_n(\omega))}{\sum_{l=1}^n \log a_{i_l^1}} = \lambda(\mathbf{p}) \text{ for } \tilde\nu_{\mathbf{p}}\text{-a.e. } \omega.$$

It follows from the definition of $\tilde\nu_{\mathbf{p}}$ that, for $\tilde\nu_{\mathbf{p}}$-a.e ω, $p_{i_l^1\ldots i_l^d} > 0$ for every l, so we may restrict our attention to these ω. If $d = 2$ then $\tilde\nu_{\mathbf{p}}(\tilde\pi B_n(\omega)) = \prod_{l=1}^n p_{i_l^1}$ and the result follows by a direct application of (KSLLN). Otherwise we have that

$$\tilde\nu_{\mathbf{p}}(\tilde\pi B_n(\omega)) = \prod_{l=1}^{L_n^{d-2}(\omega)} p_{i_l^1\ldots i_l^{d-1}} \prod_{k=1}^{d-2} \prod_{l=L_n^k(\omega)+1}^{L_n^{k-1}(\omega)} p_{i_l^1\ldots i_l^k}$$

and

$$\begin{aligned}\frac{\log \tilde\nu_{\mathbf{p}}(\tilde\pi B_n(\omega))}{\sum_{l=1}^n \log a_{i_l^1}} &= \frac{\sum_{l=1}^{L_n^{d-2}(\omega)} \log p_{i_l^1\ldots i_l^{d-1}}}{\sum_{l=1}^n \log a_{i_l^1}} \\ &\quad + \sum_{k=1}^{d-2} \frac{\sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1\ldots i_l^k}}{\sum_{l=1}^n \log a_{i_l^1}} \, \frac{\sum_{l=L_n^k(\omega)+1}^{L_n^{k-1}(\omega)} \log p_{i_l^1\ldots i_l^k}}{\sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1\ldots i_l^k}} \\ &= \gamma_n + \sum_{k=1}^{d-2} \alpha_n^k\, \beta_n^k.\end{aligned}$$

By successive application of Lemma 1 one gets that

$$\frac{L_n^{d-2}(\omega)}{n} \to \frac{\sum_{i^1} p_{i^1} \log a_{i^1}}{\sum_{i^1\ldots i^{d-1}} p_{i^1\ldots i^{d-1}} \log a_{i^1\ldots i^{d-1}}} \text{ for } \tilde\nu_{\mathbf{p}}\text{-a.e. } \omega,$$

and so, by (KSLLN),

$$\gamma_n = \frac{\frac{L_n^{d-2}(\omega)}{n} \frac{1}{L_n^{d-2}(\omega)} \sum_{l=1}^{L_n^{d-2}(\omega)} \log p_{i_l^1\ldots i_l^{d-1}}}{\frac{1}{n}\sum_{l=1}^n \log a_{i_l^1}} \to \frac{\sum_{i^1\ldots i^{d-1}} p_{i^1\ldots i^{d-1}} \log p_{i^1\ldots i^{d-1}}}{\sum_{i^1\ldots i^{d-1}} p_{i^1\ldots i^{d-1}} \log a_{i^1\ldots i^{d-1}}}$$

for $\tilde\nu_{\mathbf{p}}$-a.e. ω. By (5), $\alpha_n^k \underset{n\to\infty}{\longrightarrow} 1$ for every k. We write

$$\beta_n^k = \frac{\frac{1}{L_n^{k-1}(\omega)} \sum_{l=1}^{L_n^{k-1}(\omega)} \log p_{i_l^1\ldots i_l^k} - \frac{L_n^k(\omega)}{L_n^{k-1}(\omega)} \frac{1}{L_n^k(\omega)} \sum_{l=1}^{L_n^k(\omega)} \log p_{i_l^1\ldots i_l^k}}{\frac{1}{L_n^{k-1}(\omega)} \sum_{l=1}^{L_n^{k-1}(\omega)} \log a_{i_l^1\ldots i_l^k}}.$$

Using (KSLLN) and Lemma 1 one gets that

$$\lim_{n\to\infty} \beta_n^k = \frac{\sum_{i^1\ldots i^k} p_{i^1\ldots i^k} \log p_{i^1\ldots i^k}}{\sum_{i^1\ldots i^k} p_{i^1\ldots i^k} \log a_{i^1\ldots i^k}} - \frac{\sum_{i^1\ldots i^k} p_{i^1\ldots i^k} \log p_{i^1\ldots i^k}}{\sum_{i^1\ldots i^{k+1}} p_{i^1\ldots i^{k+1}} \log a_{i^1\ldots i^{k+1}}} \text{ for } \tilde\nu_{\mathbf{p}}\text{-a.e. } \omega,$$

and this gives what we want after a simple rearrangement. □

Lemma 3. $\dim_{\mathrm{H}} \mu_{\mathbf{p}} = \lambda(\mathbf{p}) + t(\mathbf{p})$.

Proof. As before, one is left to prove that

$$\lim_{n\to\infty} \frac{\log \tilde\mu_{\mathbf{p}}(B_n(\omega))}{\sum_{l=1}^n \log a_{i_l^1}} = \lambda(\mathbf{p}) + t(\mathbf{p}) \text{ for } \tilde\mu_{\mathbf{p}}\text{-a.e. } \omega.$$

We have that

$$\tilde{\mu}_{\mathbf{p}}(B_n(\omega)) = \tilde{\nu}_{\mathbf{p}}(\tilde{\pi} B_n(\omega)) \underbrace{\prod_{l=1}^{L_n^{d-1}(\omega)} \frac{a_{i_l^1 \dots i_l^d}^{t(\mathbf{p})}}{\sum_{i^d} a_{i_l^1 \dots i_l^{d-1} i^d}^{t(\mathbf{p})}}}_{\alpha_n},$$

By Lemma 2, we only have to prove that

$$\lim_{n\to\infty} \frac{\log \alpha_n}{\sum_{l=1}^n \log a_{i_l^1}} = t(\mathbf{p}) \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

But

$$\begin{aligned}\frac{\log \alpha_n}{\sum_{l=1}^n \log a_{i_l^1}} &= t(\mathbf{p}) \frac{\sum_{l=1}^{L_n^{d-1}(\omega)} \log a_{i_l^1 \dots i_l^d}}{\sum_{l=1}^n \log a_{i_l^1}} - \frac{\frac{1}{L_n^{d-1}(\omega)} \sum_{l=1}^{L_n^{d-1}(\omega)} \log\left(\sum_{i^d} a_{i_l^1 \dots i_l^{d-1} i^d}^{t(\mathbf{p})} \right)}{\frac{n}{L_n^{d-1}(\omega)} \frac{1}{n} \sum_{l=1}^n \log a_{i_l^1}} \\ &= t(\mathbf{p})\, \beta_n - \frac{\gamma_n}{\delta_n}.\end{aligned}$$

That $\beta_n \to 1$ follows from (5). Now we can write

$$\gamma_n = \sum_{i^1 \dots i^{d-1}} \frac{P(\omega, L_n^{d-1}(\omega), i^1 \dots i^{d-1})}{L_n^{d-1}(\omega)} \log \left(\sum_{i^d} a_{i^1 \dots i^{d-1} i^d}^{t(\mathbf{p})} \right),$$

where

$$P(\omega, n, i^1 \dots i^{d-1}) = \sharp\{1 \le l \le n : (i_l^1 \dots i_l^{d-1}) = (i^1 \dots i^{d-1})\}$$

for $(i^1, \dots, i^{d-1}) \in \mathcal{J}$. By (KSLLN),

$$\frac{P(\omega, n, i^1 \dots i^{d-1})}{n} \to p_{i^1 \dots i^{d-1}} \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega,$$

so, by the definition of $t(\mathbf{p})$,

$$\gamma_n \to 0 \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega.$$

Since $n/L_n^{d-1}(\omega) \ge 1$, we have that $|\delta| \ge \log(\min a_{i^1}^{-1}) > 0$, so we also have that

$$\frac{\gamma_n}{\delta_n} \to 0 \text{ for } \tilde{\mu}_{\mathbf{p}}\text{-a.e. } \omega,$$

thus completing the proof. □

As noticed in the beginning of this Part, these lemmas imply

$$\dim_{\mathrm{H}} \Lambda \ge \sup_{\mathbf{p}} \{\lambda(\mathbf{p}) + t(\mathbf{p})\}.$$

Part 2: $\dim_{\mathrm{H}} \Lambda \le \sup_{\mathbf{p}} \{\lambda(\mathbf{p}) + t(\mathbf{p})\}$

In this part, $d = 3$.

Given a probability vector $\mathbf{p}$ we define

$$\lambda_1(\mathbf{p}) = \frac{\sum_i p_i \log p_i}{\sum_i p_i \log c_i} \quad \text{and} \quad \lambda_2(\mathbf{p}) = \frac{\sum_{i,j} p_{ij} \log p_{ij} - \sum_i p_i \log p_i}{\sum_{i,j} p_{ij} \log b_{ij}}.$$

Let $\underline{t} = \min_{\mathbf{p}} t(\mathbf{p})$ and $\overline{t} = \max_{\mathbf{p}} t(\mathbf{p})$. Also let

$$\mathcal{P} = \left\{ \mathbf{p} = (p_{ij})_{(i,j)\in\mathcal{J}} : p_{ij} > 0 \text{ for all } (i,j) \in \mathcal{J} \text{ and } \sum_{i,j} p_{ij} = 1 \right\}.$$

Lemma 4. *Given $t \in (\underline{t}, \overline{t})$ and $\rho \in (0,1]$, there exists a probability vector $\mathbf{p} = \mathbf{p}(t,\rho)$, continuously varying, such that $t(\mathbf{p}) = t$ and*

$$p_{ij} = c_i^{\lambda_1(\mathbf{p})}\, b_{ij}^{\lambda_2(\mathbf{p})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha} \Big(\sum_j b_{ij}^{\lambda_2(\mathbf{p})} \Big(\sum_k a_{ijk}^t\Big)^{\alpha}\Big)^{\rho-1}, \quad (i,j) \in \mathcal{J}$$

where $\alpha = \alpha(t,\rho) \in \mathbb{R}$ is C^1. Moreover, $\partial\alpha/\partial t > 0$ and, for each $\rho \in (0,1]$, $\alpha(t,\rho) \to -\infty$ when $t \to \underline{t}$ and $\alpha(t,\rho) \to \infty$ when $t \to \overline{t}$.

Proof. Given $\alpha, \lambda_1, \lambda_2 \in \mathbb{R}$, $t \in (\underline{t}, \overline{t})$ and $\rho \in (0,1]$, we define a probability vector $\mathbf{p}(\alpha, \lambda_1, \lambda_2, t, \rho)$ by

$$p_{ij}(\alpha, \lambda_1, \lambda_2, t, \rho) = C(\alpha, \lambda_1, \lambda_2, t, \rho)\, c_i^{\lambda_1}\, b_{ij}^{\lambda_2} \Big(\sum_k a_{ijk}^t\Big)^{\alpha} \gamma_i(\alpha, \lambda_2, t)^{\rho-1} \tag{9}$$

where

$$\gamma_i(\alpha, \lambda_2, t) = \sum_j b_{ij}^{\lambda_2} \Big(\sum_k a_{ijk}^t\Big)^{\alpha}$$

and

$$C(\alpha, \lambda_1, \lambda_2, t, \rho) = \Big(\sum_i c_i^{\lambda_1}\, \gamma_i(\alpha, \lambda_2, t)^{\rho}\Big)^{-1},$$

for each $(i,j) \in \mathcal{J}$.

Let F be the continuous function defined by

$$F(\alpha, \lambda_1, \lambda_2, t, \rho) = \sum_{i,j} p_{ij}(\alpha, \lambda_1, \lambda_2, t, \rho) \log\Big(\sum_k a_{ijk}^t\Big).$$

We are going to prove there exists a unique $\alpha = \alpha(\lambda_1, \lambda_2, t, \rho)$, continuously varying, such that $F(\alpha, \lambda_1, \lambda_2, t, \rho) = 0$, i.e. $t(\mathbf{p}(\alpha, \lambda_1, \lambda_2, t, \rho)) = t$.

Unicity. We have that, for each $(i,j) \in \mathcal{J}$,

$$\frac{\partial p_{ij}}{\partial \alpha} = \frac{1}{C}\frac{\partial C}{\partial \alpha} p_{ij} + \log\Big(\sum_j a_{ijk}^t\Big) p_{ij} + (\rho - 1)\frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \alpha} p_{ij}.$$

Also,

$$\frac{1}{C}\frac{\partial C}{\partial \alpha} = -\rho \sum_i p_i \frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \alpha} \tag{10}$$

and

$$\frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \alpha} = \sum_j \frac{p_{ij}}{p_i} \log\Big(\sum_k a_{ijk}^t\Big) \tag{11}$$

where

$$p_i = \sum_j p_{ij} = C c_i^{\lambda_1} \gamma_i^{\rho}.$$

So, by simple rearrangement we get

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \sum_{i,j} \frac{\partial p_{ij}}{\partial \alpha} \log\Big(\sum_k a_{ijk}^t\Big) \\ &= \rho \left\{ \sum_i p_i \Big(\sum_j \frac{p_{ij}}{p_i} \log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 - \Big(\sum_{i,j} p_{ij} \log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 \right\} \\ &\quad + \sum_i p_i \left\{ \sum_j \frac{p_{ij}}{p_i} \Big(\log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 - \Big(\sum_j \frac{p_{ij}}{p_i} \log\Big(\sum_k a_{ijk}^t\Big)\Big)^2 \right\}. \end{aligned}$$

By the Cauchy-Schwarz inequality we have that the expressions between curly brackets are non-negative and the second one is positive if there exists $i \in \{1, ..., m\}$ such that the function

$$j \mapsto \sum_k a_{ijk}^t$$

is non-constant (note that $\mathbf{p} \in \mathcal{P}$). This is guaranteed by hypothesis (1). Thus $\partial F/\partial\alpha > 0$.

Existence. For fixed $(\lambda_1, \lambda_2, t, \rho)$, we will look at the limit distributions of $\mathbf{p}(\alpha) = \mathbf{p}(\alpha, \lambda_1, \lambda_2, t, \rho)$ as α goes to $+\infty$ and $-\infty$. For $(i,j) \in \mathcal{J}$, define t_{ij} to be the unique real in $[0,1]$ satisfying

$$\sum_k a_{ijk}^{t_{ij}} = 1.$$

It is easy to see that $\underline{t} = \min_{(i,j)\in\mathcal{J}} t_{ij}$ and $\overline{t} = \max_{(i,j)\in\mathcal{J}} t_{ij}$. Let

$$A_t = \max_{(i,j)\in\mathcal{J}} \sum_k a_{ijk}^t.$$

For $(i,j) \in \mathcal{J}$ such that $t < t_{ij}$ we have that

$$\sum_k a_{ijk}^t > \sum_k a_{ijk}^{t_{ij}} = 1,$$

so $A_t > 1$. Consider $(\bar{i}, \bar{j}), (i,j) \in \mathcal{J}$ such that

$$\sum_k a_{ijk}^t < \sum_k a_{\bar{i}\bar{j}k}^t = A_t.$$

We have

$$\frac{p_{ij}(\alpha)}{p_{\bar{i}\bar{j}}(\alpha)} \leq C \left(\frac{\sum_k a_{ijk}^t}{\sum_k a_{\bar{i}\bar{j}k}^t} \right)^{\alpha} \left(\frac{\gamma_{\bar{i}}(\alpha)}{\gamma_i(\alpha)} \right)^{1-\rho},$$

for some constant C not depending on α. Now, for $\alpha > 0$,

$$\frac{\gamma_{\bar{i}}(\alpha)}{\gamma_i(\alpha)} \leq \tilde{C} \left(\frac{\sum_k a_{\bar{i}\bar{j}k}^t}{\sum_k a_{ijk}^t} \right)^{\alpha},$$

for some constant $\tilde{C}$ not depending on α. So,

$$\frac{p_{ij}(\alpha)}{p_{\bar{i}\bar{j}}(\alpha)} \leq C\tilde{C} \left(\frac{\sum_k a_{ijk}^t}{\sum_k a_{\bar{i}\bar{j}k}^t} \right)^{\alpha\rho},$$

which converges to 0 as $\alpha \to \infty$. This implies that

$$\sum_{i,j} p_{ij}(\alpha) \log\Big(\sum_k a_{ijk}^t\Big) \underset{\alpha\to\infty}{\longrightarrow} \log A_t > 0. \tag{12}$$

In the same way, defining

$$B_t = \min_{(i,j)\in\mathcal{J}} \sum_k a_{ijk}^t < 1,$$

and taking $(\underline{i}, \underline{j}), (i,j) \in \mathcal{J}$ such that

$$B_t = \sum_k a_{\underline{i}\underline{j}k}^t < \sum_k a_{ijk}^t,$$

we get, for $\alpha < 0$,

$$\frac{p_{ij}(\alpha)}{p_{\underline{i}\underline{j}}(\alpha)} \leq C\tilde{C} \left(\frac{\sum_k a_{ijk}^t}{\sum_k a_{\underline{i}\underline{j}k}^t} \right)^{\alpha\rho},$$

which converges to 0 as $\alpha \to -\infty$. This implies that

$$\sum_{i,j} p_{ij}(\alpha) \log\Big(\sum_k a_{ijk}^t\Big) \underset{\alpha\to-\infty}{\longrightarrow} \log B_t < 0. \tag{13}$$

By (12), (13) and continuity, there exists $\alpha \in \mathbb{R}$ such that $F(\alpha, \lambda_1, \lambda_2, t, \rho) = 0$. The continuity of $\alpha(\lambda_1, \lambda_2, t, \rho)$ follows from the uniqueness part and the implicit function theorem. Actually, since $F(\alpha, \lambda_1, \lambda_2, t, \rho)$ is continuously differentiable, we also get that $\alpha(\lambda_1, \lambda_2, t, \rho)$ is continuoulsy differentiable. Moreover,

$$\frac{\partial \alpha}{\partial t} = -\left(\frac{\partial F}{\partial \alpha}\right)^{-1} \frac{\partial F}{\partial t}$$

where

$$\frac{\partial F}{\partial t} = \sum_{i,j} p_{ij} \frac{\sum_k a_{ijk}^t}{\sum_k a_{ijk}^t \log a_{ijk}}$$

which satisfies

$$-\infty < (\log \max a_{ijk})^{-1} \le \frac{\partial F}{\partial t} \le (\log \min a_{ijk})^{-1} < 0,$$

and so $\partial\alpha/\partial t > 0$. Observe that $t(\mathbf{p}) = \overline{t} \Rightarrow \mathbf{p} \in \partial\mathcal{P}$ (in this lemma we are assuming $\underline{t} < \overline{t}$), so since

$$t(\mathbf{p}(\alpha(\lambda_1, \lambda_2, t, \rho))) \to \overline{t} \quad \text{when} \quad t \to \overline{t}$$

then

$$\mathbf{p}(\alpha(\lambda_1, \lambda_2, t, \rho)) \to \partial\mathcal{P} \quad \text{when} \quad t \to \overline{t},$$

which implies

$$\alpha(\lambda_1, \lambda_2, t, \rho) \to \infty \quad \text{when} \quad t \to \overline{t}$$

(this convergence is uniform in $\lambda_1, \lambda_2 \in [0, 1]$). In the same way we see that

$$\alpha(\lambda_1, \lambda_2, t, \rho) \to -\infty \quad \text{when} \quad t \to \underline{t}.$$

Now we want to find $\lambda_1 = \lambda_1(\lambda_2, t, \rho)$ differentiable such that

$$C(\alpha(\lambda_1, \lambda_2, t, \rho), \lambda_1, \lambda_2, t, \rho) = 1. \tag{14}$$

We have

$$\frac{\partial}{\partial \lambda_1} C(\alpha(\lambda_1, \lambda_2, t, \rho), \lambda_1, \lambda_2, t, \rho) = \frac{\partial C}{\partial \alpha}\frac{\partial \alpha}{\partial \lambda_1} + \frac{\partial C}{\partial \lambda_1}$$

Observe that, by (10) and (11),

$$\frac{\partial C}{\partial \alpha} = 0$$

(at points $(\alpha(\lambda_1, \lambda_2, t, \rho), \lambda_1, \lambda_2, t, \rho)$), and

$$\frac{\partial \log C}{\partial \lambda_1} = \frac{1}{C}\frac{\partial C}{\partial \lambda_1} = -\sum_i p_i \log c_i \ge \min_i \log c_i^{-1} > 0. \tag{15}$$

So, $C(\alpha(\lambda_1, \lambda_2, t, \rho), \lambda_1, \lambda_2, t, \rho)$ is a differentiable function, for each (λ_2, t, ρ), is strictly increasing in λ_1 and, by (15), has limit ∞ as $\lambda_1 \to \infty$ and limit 0 as $\lambda_1 \to -\infty$. By the implicit function theorem, there is a unique $\lambda_1 = \lambda_1(\lambda_2, t, \rho)$, which is differentiable, satisfying (14). Moreover,

$$\frac{\partial \lambda_1}{\partial \lambda_2} = -\left(\frac{\partial C}{\partial \lambda_1}\right)^{-1} \frac{\partial C}{\partial \lambda_2} \tag{16}$$

and

$$\frac{\partial C}{\partial \lambda_2} = -\rho C \sum_i p_i \frac{1}{\gamma_i}\frac{\partial \gamma_i}{\partial \lambda_2}. \tag{17}$$

Now we see that

$$\frac{1}{\gamma_i}\frac{\partial\gamma_i}{\partial\lambda_2}=\sum_j\frac{p_{ij}}{p_i}\log b_{ij}. \tag{18}$$

So, by (15)-(18) we get

$$\frac{\partial\lambda_1}{\partial\lambda_2}=-\rho\frac{\sum_{i,j}p_{ij}\log b_{ij}}{\sum_i p_i\log c_i} \tag{19}$$

We use the following notation

$$\tilde\Theta(\lambda_2,t,\rho)=(\alpha(\lambda_1(\lambda_2,t,\rho),\lambda_2,t,\rho),\lambda_1(\lambda_2,t,\rho),\lambda_2,t,\rho),$$
$$\Theta(\lambda_2,t,\rho)=(\alpha(\lambda_1(\lambda_2,t,\rho),\lambda_2,t,\rho),\lambda_2,t,\rho).$$

We see that

$$\lambda_1(\mathbf{p}(\tilde\Theta))=\lambda_1+\rho\,\frac{\sum_i p_i(\tilde\Theta)\log\gamma_i(\Theta)}{\sum_i p_i(\tilde\Theta)\log c_i},$$
$$\lambda_2(\mathbf{p}(\tilde\Theta))=\lambda_2-\frac{\sum_i p_i(\tilde\Theta)\log\gamma_i(\Theta)}{\sum_{i,j}p_{ij}(\tilde\Theta)\log b_{ij}}.$$

So, we are left to prove there exists $\lambda_2=\lambda_2(t,\rho)$, continuously varying, such that

$$\sum_i p_i(\tilde\Theta)\log\gamma_i(\Theta)=0. \tag{20}$$

We have that

$$\frac{\partial}{\partial\lambda_2}\sum_i p_i(\tilde\Theta)\log\gamma_i(\Theta)=\frac{\partial\lambda_1}{\partial\lambda_2}\sum_i p_i(\tilde\Theta)\log c_i+(\rho+1)\sum_i p_i(\tilde\Theta)\frac{1}{\gamma_i(\Theta)}\frac{\partial}{\partial\lambda_2}\gamma_i(\Theta)$$
$$=-\rho\sum_{i,j}p_{ij}(\tilde\Theta)\log b_{ij}+(\rho+1)\sum_i p_i(\tilde\Theta)\frac{1}{\gamma_i(\Theta)}\frac{\partial}{\partial\lambda_2}\gamma_i(\Theta), \tag{21}$$

where we have used (19). Now we see that

$$\frac{1}{\gamma_i(\Theta)}\frac{\partial}{\partial\lambda_2}\gamma_i(\Theta)=\frac{1}{\gamma_i(\Theta)}\frac{\partial\gamma_i}{\partial\alpha}(\Theta)\left(\frac{\partial\alpha}{\partial\lambda_1}\frac{\partial\lambda_1}{\lambda_2}+\frac{\partial\alpha}{\partial\lambda_2}\right)+\frac{1}{\gamma_i(\Theta)}\frac{\partial\gamma_i}{\partial\lambda_2}(\Theta) \tag{22}$$

and

$$\frac{1}{\gamma_i(\Theta)}\frac{\partial\gamma_i}{\partial\lambda_2}(\Theta)=\sum_j\frac{p_{ij}(\tilde\Theta)}{p_i(\tilde\Theta)}\log b_{ij}. \tag{23}$$

Also remember from (11) that

$$\sum_i p_i(\tilde\Theta)\frac{1}{\gamma_i(\Theta)}\frac{\partial\gamma_i}{\partial\alpha}(\Theta)=0. \tag{24}$$

Then, it follows from (21)-(24) that

$$\frac{\partial}{\partial\lambda_2}\sum_i p_i(\tilde\Theta)\log\gamma_i(\Theta)=\sum_{i,j}p_{ij}(\tilde\Theta)\log b_{ij}\le\max_{i,j}\log b_{ij}<0,$$

and, as before, by the implicit function theorem, this implies there exists a unique $\lambda_2=\lambda_2(t,\rho)$, continuously varying, satisfying (20), thus concluding the proof of Lemma 4. □

Lemma 5. *For every $\omega\in\Omega$, $n\in\mathbb{N}$ and $k=1,2$,*

$$L^k_{n+1}(\omega)-L^k_n(\omega)\le 1.$$

Proof. We have that, for $k=1$,

$$\prod_{l=1}^{n+1} c_{i_l} \le \prod_{l=1}^{L^1_{n+1}(\omega)} b_{i_l j_l} \quad \text{and} \quad \prod_{l=1}^{n} c_{i_l} > \prod_{l=1}^{L^1_n(\omega)+1} b_{i_l j_l}$$

so

$$\prod_{l=L^1_n(\omega)+2}^{L^1_{n+1}(\omega)} b_{i_l j_l} > c_{i_{n+1}}$$

which implies

$$L^1_{n+1}(\omega) - L^1_n(\omega) \le \frac{\log \min c_i}{\log \max b_{ij}} + 1 < 2.$$

The case $k=2$ is similar. □

Let $s = \sup_{\mathbf{p}}\{\lambda(\mathbf{p}) + t(\mathbf{p})\}$.

Lemma 6. *For every $\omega \in \Omega$ there exists $\mathbf{p} \in \mathcal{P}$ such that*

$$\liminf_{n\to\infty} \frac{\log \tilde{\mu}_{\mathbf{p}}(B_n(\omega))}{\sum_{l=1}^n \log c_{i_l}} \le s.$$

Proof. Fix $\omega \in \Omega$. We use the notation

$$d_{\mathbf{p},n}(\omega) = \frac{\log \tilde{\mu}_{\mathbf{p}}(B_n(\omega))}{\sum_{l=1}^n \log c_{i_l}}.$$

Then it follows from the proofs of Lemma 2 and Lemma 3 that, if $\mathbf{p} \in \mathcal{P}$,

$$\begin{aligned} d_{\mathbf{p},n}(\omega) =& \frac{\sum_{l=1}^n \log p_{i_l}}{\sum_{l=1}^n \log c_{i_l}} + \beta_n(\omega)\, \frac{\sum_{l=1}^{L^1_n(\omega)} \log p_{i_l j_l} - \sum_{l=1}^{L^1_n(\omega)} \log p_{i_l}}{\sum_{l=1}^{L^1_n(\omega)} \log b_{i_l j_l}} \\ &+ \eta_n(\omega) t(\mathbf{p}) - \frac{\sum_{l=1}^{L^2_n(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t(\mathbf{p})}\Big)}{\sum_{l=1}^n \log c_{i_l}} \end{aligned} \tag{25}$$

where, by (5),

$$\beta_n(\omega) = \frac{\sum_{l=1}^{L^1_n(\omega)} \log b_{i_l j_l}}{\sum_{l=1}^n \log c_{i_l}} \underset{n\to\infty}{\longrightarrow} 1$$

and

$$\eta_n(\omega) = \frac{\sum_{l=1}^{L^2_n(\omega)} \log a_{i_l j_l k_l}}{\sum_{l=1}^n \log c_{i_l}} \underset{n\to\infty}{\longrightarrow} 1.$$

Given $t \in (\underline{t}, \overline{t})$ and $\rho \in (0,1]$, consider the probability vector $\mathbf{p}(t,\rho)$, such that $t(\mathbf{p}(t,\rho)) = t$, given by Lemma 4. Applying (25) to $\mathbf{p}(t,\rho)$ we obtain

$$\begin{aligned} d_{\mathbf{p}(t,\rho),n}(\omega) &= \lambda_1(\mathbf{p}(t,\rho)) + \beta_n(\omega)\lambda_2(\mathbf{p}(t,\rho)) + \eta_n(\omega) t \\ &+ \frac{\rho \sum_{l=1}^n \log \gamma_{i_l}(t,\rho) - \sum_{l=1}^{L^1_n(\omega)} \log \gamma_{i_l}(t,\rho)}{\sum_{l=1}^n \log c_{i_l}} \\ &+ \frac{\alpha(t,\rho) \sum_{l=1}^{L^1_n(\omega)} \log\Big(\sum_k a^t_{i_l j_l k}\Big) - \sum_{l=1}^{L^2_n(\omega)} \log\Big(\sum_k a^t_{i_l j_l k}\Big)}{\sum_{l=1}^n \log c_{i_l}}, \end{aligned} \tag{26}$$

where

$$\gamma_i(t,\rho) = \sum_j b_{ij}^{\lambda_2(\mathbf{p}(t,\rho))} \Big(\sum_k a^t_{ijk}\Big)^{\alpha(t,\rho)}.$$

So we must prove that there exist $t_* \in (\underline{t}, \overline{t})$ and $\rho_* \in (0,1]$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\Big\{\rho_* \sum_{l=1}^{n} \log\gamma_{i_l}(t_*,\rho_*) - \sum_{l=1}^{L_n^1(\omega)} \log\gamma_{i_l}(t_*,\rho_*) \tag{27}$$
$$+ \alpha(t_*,\rho_*) \sum_{l=1}^{L_n^1(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t_*}\Big) - \sum_{l=1}^{L_n^2(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t_*}\Big)\Big\} \geq 0.$$

By Lemma 4 and the inverse function theorem, given $\rho \in (0,1]$ and $a \in \mathbb{R}$, there exists a unique function $t(\rho, a) \in (\underline{t}, \overline{t})$, which is continuous, increasing in a and satisfies

$$\alpha(t(\rho,a),\rho) = a. \tag{28}$$

Let

$$\rho_0 = \liminf_{n\to\infty} \frac{L_n^1(\omega)}{n}, \quad \rho_1 = \limsup_{n\to\infty} \frac{L_n^1(\omega)}{n},$$
$$a_0 = \liminf_{n\to\infty} \frac{L_n^2(\omega)}{L_n^1(\omega)}, \quad a_1 = \limsup_{n\to\infty} \frac{L_n^2(\omega)}{L_n^1(\omega)},$$

and

$$t_0 = \min_{\rho\in[\rho_0,\rho_1]} t(\rho,a_0), \quad t_1 = \max_{\rho\in[\rho_0,\rho_1]} t(\rho,a_1).$$

Let

$$\rho_n = \frac{L_n^1(\omega)}{n}, \quad a_n = \frac{L_n^2(\omega)}{L_n^1(\omega)} \quad \text{and} \quad t_n = t(\rho_n, a_n).$$

Note that, using Lemma 5, we easily see that (for all n sufficiently large)

$$|\rho_{n+1} - \rho_n| \leq \frac{2}{n} \quad \text{and} \quad |a_{n+1} - a_n| \leq \frac{4\tau^{-2}}{n}$$

where

$$\tau = \frac{\log \max c_i}{2\log \min a_{ijk}},$$

so we satisfy Lemma 8 hypotheses (see also Corollary 1 and Remark 2). Then, by Lemma 8, for every $(t,\rho) \in [t_0,t_1]\times[\rho_0,\rho_1]$

$$F(t,\rho) = \limsup_{n\to\infty} \frac{1}{n}\Big\{\rho_n \sum_{l=1}^{n} \log\gamma_{i_l}(t,\rho) - \sum_{l=1}^{L_n^1(\omega)} \log\gamma_{i_l}(t,\rho) \tag{29}$$
$$+ a_n \sum_{l=1}^{L_n^1(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t}\Big) - \sum_{l=1}^{L_n^2(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t}\Big)\Big\} \geq 0.$$

Since, for all (i,j),

$$(t,\rho) \mapsto \log\gamma_i(t,\rho), \quad t \mapsto \log\Big(\sum_k a_{ijk}^t\Big) \tag{30}$$

are continuous functions, so is $F(t,\rho)$. By adding some constant, we may assume the functions in (30) are ≥ 1, because by definition of ρ_n and a_n this does not change $F(t,\rho)$. Let $\bar\rho(t,\rho)$ be the biggest accumulation point of (ρ_n) for which the $\limsup$ in (29) is attained, so that

$$F(t,\rho) = \limsup_{n\to\infty} \frac{1}{n}\Big\{\bar\rho(t,\rho) \sum_{l=1}^{n} \log\gamma_{i_l}(t,\rho) - \sum_{l=1}^{L_n^1(\omega)} \log\gamma_{i_l}(t,\rho)$$
$$+ a_n \sum_{l=1}^{L_n^1(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t}\Big) - \sum_{l=1}^{L_n^2(\omega)} \log\Big(\sum_k a_{i_l j_l k}^{t}\Big)\Big\}.$$

The continuity of F and the functions in (30) imply that $\bar\rho(t,\rho)$ is also continuous. By (28) we can also write

$$F(t,\rho) = \limsup_{n\to\infty} \frac{1}{n}\Big\{\bar\rho(t,\rho)\sum_{l=1}^{n}\log\gamma_{i_l}(t,\rho) - \sum_{l=1}^{L_n^1(\omega)}\log\gamma_{i_l}(t,\rho) \\ + \alpha(t_n,\bar\rho(t,\rho))\sum_{l=1}^{L_n^1(\omega)}\log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n^2(\omega)}\log\Big(\sum_k a_{i_l j_l k}^t\Big)\Big\}. \tag{31}$$

Now let $\bar t(t,\rho)$ be the biggest accumulation point of (t_n) for which the lim sup in (31) is attained. Using the same arguments as before we see that $\bar t(t,\rho)$ is continuous and

$$F(t,\rho) = \limsup_{n\to\infty} \frac{1}{n}\Big\{\bar\rho(t,\rho)\sum_{l=1}^{n}\log\gamma_{i_l}(t,\rho) - \sum_{l=1}^{L_n^1(\omega)}\log\gamma_{i_l}(t,\rho) \\ + \alpha(\bar t(t,\rho),\bar\rho(t,\rho))\sum_{l=1}^{L_n^1(\omega)}\log\Big(\sum_k a_{i_l j_l k}^t\Big) - \sum_{l=1}^{L_n^2(\omega)}\log\Big(\sum_k a_{i_l j_l k}^t\Big)\Big\}.$$

Since

$$(\bar t,\bar\rho)\colon [t_0,t_1]\times[\rho_0,\rho_1]\to[t_0,t_1]\times[\rho_0,\rho_1]$$

is continuous, by Brouwer's fixed point theorem there is $(t_*,\rho_*)\in[t_0,t_1]\times[\rho_0,\rho_1]$ such that

$$\bar t(t_*,\rho_*) = t_* \quad\text{and}\quad \bar\rho(t_*,\rho_*) = \rho_*$$

thus proving (27). □

Part 2 will be concluded in the following lemma.

Lemma 7.

$$\dim_{\mathrm H}\Lambda \le \sup_{\mathbf p}\{\lambda(\mathbf p)+t(\mathbf p)\}.$$

Proof. Let $\varepsilon>0$. Consider the *approximate cubes of order* n given by $B_n(z) = \chi(B_n(\omega))$ where $\omega\in\chi^{-1}(z)$, $z\in\Lambda$, $n\in\mathbb N$. Then it follows from Lemma 6 that

$$\forall_{z\in\Lambda}\ \forall_{N\in\mathbb N}\ \exists_{n>N}\ \exists_{\mathbf p\in\mathcal P}:\ \frac{\log\mu_{\mathbf p}(B_n(z))}{\log|B_n(z)|}\le s+\varepsilon. \tag{32}$$

Given $\delta,\eta>0$, we shall build a cover $\mathcal U_{\delta,\eta}$ of Λ by sets with diameter $<\eta$ such that

$$\sum_{U\in\mathcal U_{\delta,\eta}}|U|^{s+\varepsilon+2\delta}\le\sqrt3\,(\max a_{ijk}^{-1})\,M_\delta$$

where M_δ is an integer depending on δ but not on η. This implies that $\dim_{\mathrm H}\Lambda\le\varepsilon+2\delta$ which gives what we want because ε and δ can be taken arbitrarily small. Let $c=\max c_i<1$. It is clear that there exists a finite number of Bernoulli measures $\mu_1,\dots,\mu_{M_\delta}$ such that

$$\forall_{\mathbf p}\ \exists_{k\in\{1,\dots,M_\delta\}}:\ \frac{\mu_{\mathbf p}(B_n)}{\mu_k(B_n)}\le c^{-\delta n}$$

for all approximate cubes of order n, B_n. By (32), we can build a cover of Λ by approximate cubes $B_{n(z^i)}$, $i=1,2,\dots$ that are disjoint and have diameters $<\eta$, such that

$$\mu_{\mathbf p^i}(B_{n(z^i)})\ge|B_{n(z^i)}|^{s+\varepsilon+\delta}$$

for some probabilty vectors $\mathbf{p}^i$. It follows that

$$\begin{aligned}\sum_i |B_{n(z^i)}|^{s+\varepsilon+2\delta} &\le \sum_i \mu_{\mathbf{p}^i}(B_{n(z^i)})\,|B_{n(z^i)}|^{\delta}\\ &\le \sum_i \mu_{k_i}(B_{n(z^i)})\,c^{-\delta n(z^i)}\,\sqrt{3}\,(\max a_{ijk}^{-1})\,c^{\delta n(z^i)}\\ &\le \sqrt{3}\,(\max a_{ijk}^{-1})\sum_{k=1}^{M_\delta}\sum_i \nu_k(B_{n(z^i)}) \le \sqrt{3}\,(\max a_{ijk}^{-1})\,M_\delta\end{aligned}$$

as we wish. □

This ends the proof of Theorem A.

4. A calculus lemma

Here we prove a calculus lemma which is a non-linear extension of [7, Lemma 4.1], and is crucial in proving Lemma 6.

Lemma 8. *Let $f_k\colon (0,\infty) \to \mathbb{R}$ be Lipschitz functions for $k = 1, 2, ..., r$, and suppose $\alpha_k\colon (0,\infty) \to \mathbb{R}$ is bounded,* C^1 *and there exist positive constants δ, C such that*

$$\bullet\ \alpha_k(u) > \delta, \tag{33}$$

$$\bullet\ |\alpha_k'(u)| \le \frac{C}{u} \tag{34}$$

for every $u > 0$. Then

$$\limsup_{u\to\infty} \frac{1}{u}\sum_{k=1}^{r}\Big(\alpha_k(u)f_k(u) - f_k(\alpha_k(u)u)\Big) \ge 0. \tag{35}$$

Proof. First we begin by proving the simpler case for which

$$|\alpha_k'(u)| \le \frac{C}{u^\sigma} \tag{36}$$

for some $\sigma > 1$, and then say how it extends to the the case (34).

Following [7], we define $g_k\colon (0,\infty) \to \mathbb{R}$ by $g_k(x) = e^{-x}f_k(e^x)$ for $k = 1, ..., r$. Then we must see that

$$\limsup_{x\to\infty}\sum_{k=1}^{r}\alpha_k(e^x)\Big(g_k(x) - g_k(x+\log\alpha_k(e^x))\Big) \ge 0. \tag{37}$$

We will see that

$$\left|\int_0^u \sum_{k=1}^{r}\alpha_k(e^x)\Big(g_k(x) - g_k(x+\log\alpha_k(e^x))\Big)\,dx\right| \tag{38}$$

is bounded in u, which implies (37).

We will use the change of coordinates $y = x + \log\alpha_k(e^x)$, which is invertible for $x > a$, for some $a > 0$, due to conditions (33) and (36). Also, the functions g_k are bounded because the functions f_k are Lipschitz. Then we find some $M > 0$ such that (38) is bounded by

$$M + \left|\int_a^u \sum_{k=1}^{r}\alpha_k(e^x)\Big(g_k(x) - g_k(x+\log\alpha_k(e^x))\Big)\,dx\right|.$$

Using the intermediate value theorem and (36) we obtain

$$\left|\sum_{k=1}^{r}\int_a^u \Big(\alpha_k(e^x) - \alpha_k(e^{x+\log\alpha_k(e^x)})\Big)g_k(x+\log\alpha_k(e^x))\,dx\right| \le M$$

for $u > 0$ (by increasing M if necessary). So, (38) is bounded by

$$2M + \left| \int_a^u \sum_{k=1}^r \Big(\alpha_k(e^x) g_k(x) - \alpha_k(e^{x+\log \alpha_k(e^x)}) g_k(x + \log \alpha_k(e^x)) \Big) \, dx \right|. \tag{39}$$

By doing the change of coordinates $y = x + \log \alpha_k(e^x)$, (39) becomes

$$2M+ \left| \sum_{k=1}^r \left(\int_a^u \alpha_k(e^x) g_k(x) \, dx - \int_{a+\log \alpha_k(e^a)}^{u+\log \alpha_k(e^u)} \alpha_k(e^y) g_k(y) \frac{\alpha_k(e^{x(y)})}{\alpha_k(e^{x(y)}) + \alpha_k'(e^{x(y)}) e^{x(y)}} \, dy \right) \right|$$

$$\leq 2M$$

$$+ \sum_{k=1}^r \left| \int_a^{a+\log \alpha_k(e^a)} \alpha_k(e^x) g_k(x) \, dx \right| + \sum_{k=1}^r \left| \int_{u+\log \alpha_k(e^u)}^u \alpha_k(e^x) g_k(x) \, dx \right| \tag{40}$$

$$+ \sum_{k=1}^r \left| \int_{a+\log \alpha_k(e^a)}^{u+\log \alpha_k(e^u)} \alpha_k(e^y) g_k(y) \frac{\alpha_k'(e^{x(y)}) e^{x(y)}}{\alpha_k(e^{x(y)}) + \alpha_k'(e^{x(y)}) e^{x(y)}} \, dy \right|. \tag{41}$$

The terms in (40) are bounded because the functions g_k and α_k are bounded. To prove the boundeness of the term in (41) we also use condition (36) (and the inverse change of coordinates). Thus (38) is bounded, concluding the proof of this first case.

Before going to the general case we notice that the proof above also works when

$$|\alpha_k'(u) u| \leq \frac{C}{\log u},$$

$u > 0$, for some constant $C > 0$, because we are allowed to change the limits of integration in (38), namely $0 < a(u) < b(u)$ such that $b(u) - a(u) \to \infty$ when $u \to \infty$, and such that

$$\int_{e^{a(u)}}^{e^{b(u)}} |\alpha_k'(x)| \, dx$$

is bounded in u. For instance,

$$a(u) = u \quad \text{and} \quad b(u) = 2u$$

will do.

Now suppose the functions α_k only satisfy (34). We define $\beta_k \colon (1, \infty) \to \mathbb{R}$ by

$$\beta_k(u) = \frac{1}{\log u} \int_1^u \frac{\alpha_k(x)}{x} \, dx.$$

Then this lemma's hypotheses are also satisfied for the functions β_k, and moreover

$$\beta_k'(u) u = \frac{\alpha_k(u) - \beta_k(u)}{\log u},$$

so, by the proof above, the result (35) holds with the functions β_k. Let (u_n) be a sequence at which the limit in (35) is attained. By taking a subsequence if necessary, we may assume that $(\beta_k(u_n))$ and $(\alpha_k(u_n))$ are convergent, $k = 1, ..., r$. We want to see that

$$|\beta_k(u_n) - \alpha_k(u_n)| \to 0 \quad \text{when} \quad n \to \infty, \tag{42}$$

because then, using that the functions f_k are Lipschitz, we also have that (35) holds with the functions α_k. By Cauchy's rule ($\lim f/g = \lim f'/g'$),

$$\lim_{n\to\infty} \beta_k(u_n) = \lim_{n\to\infty} \frac{\int_1^{u_n} \frac{\alpha_k(x)}{x} \, dx}{\log u_n} = \lim_{n\to\infty} \alpha_k(u_n),$$

as we want (note that we can realise $u_n = \xi(n)$ where $\xi(x)$ is a differentiable function). $\square$

Corollary 1. *Suppose in addition to Lemma 8 hypotheses there are functions* $\beta_k\colon (0,\infty) \to \mathbb{R}$, $k = 1, ..., r$ *satisfying the same hypotheses of* α_k. *Then*

$$\limsup_{u\to\infty} \frac{1}{u}\sum_{k=1}^{r}\Big(\frac{\alpha_k(u)}{\beta_k(u)}f_k(\beta_k(u)u) - f_k(\alpha_k(u)u)\Big) \geq 0.$$

Proof. Define $g_k\colon (0,\infty) \to \mathbb{R}$ by $g_k(u) = f_k(\beta_k(u)u)$. Since the functions β_k satisfy the same conditions (33) and (34) as α_k, we can easily see that the functions $\frac{\alpha_k}{\beta_k}$ also do satisfy them, and that g_k are Lipschitz functions. Since

$$\frac{\alpha_k(u)}{\beta_k(u)}f_k(\beta_k(u)u) - f_k(\alpha_k(u)u) = \frac{\alpha_k(u)}{\beta_k(u)}g_k(u) - g_k(\frac{\alpha_k(u)}{\beta_k(u)}u),$$

we can apply Lemma 8. □

Remark 2. Lemma 8 (and its Corollary 1) also works when the functions f_k and α_k are defined only on the positive integers, by extending them in a piecewise linear fashion, if we substitute f_k being Lipschitz and hypothesis (34) on α_k by, respectively,

$$|f_k(n+1) - f_k(n)| \leq C \quad \text{and} \quad |\alpha_k(n+1) - \alpha_k(n)| \leq \frac{C}{n}$$

for all n, for some constant $C > 0$.

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